On Profinite Structures

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Abstract

We prove that first-order profinite structures, i.e., the projective limits of downward directed systems of finite discrete structures, are pure injective, that is, they possess the extension property relative to the class of pure embeddings. We present a topological characterization of the notion of profinite structure which yields closure properties of this class of structures and a description of its quotients objects, generalizing results in [**Lim**] and [**Mrn**]. We discuss an interesting elementary class of first-order topological structures and introduce the notion of a saturated family of congruences, associating to each such pair of data a functor, called Profinite Hull. It is shown, by two different approaches that this functor satisfies an universal property, yielding a pair of adjoint functors. It is also established that the profinite hull functor preserves inductive limits and quotients by a saturated congruence. It is considered a "local-global principle" naturally associated to the profinite hull functor.

Introduction

If L is a first-order language with equality, write **L-mod** for the category of L-structures and L-morphisms. The language L (although arbitrary) will remain fixed in all that follows.

We shall consider profinite L-structures, i.e. the projective limits of downward directed systems of finite discrete L-structures. The study of the theory of profinite L-structures has received some attention, being explicitly mentioned in the European program of research in Model Theory *MODNET* (item 3 of Task I: Pure Model Theory).

This work is a sequence to [**MM2**], where it is shown that profinite L-structures are retracts of certain ultraproducts of finite L-structures and, as a consequence, any elementary class $\mathcal{A}$ of **L-mod** that is axiomatizable by sentences of the form $\forall\vec{x}(\psi_0(\vec{x}) \rightarrow \psi_1(\vec{x}))$, where $\psi_0(\vec{x}), \psi_1(\vec{x})$ are positive-existential L-formulas, is closed under profinite limits. These considerations apply, in particular, to *Special Groups* and *Reduced Special Groups*, a first-order axiomatization of the algebraic theory of quadratic forms and its reduced counterpart, respectively. The reader is referred to [**DM2**] for a presentation of this circle of ideas, including the appropriate first-order language. Most of the results proved here generalize results first obtained for reduced special groups (see [**Mrn**]) and for varieties of algebras. In particular, they also apply to Boolean algebras, that, in a sense that can be made precise (see Chapter 4 in [**DM2**]), is a subcategory of the category of reduced special groups.

The paper contains four sections. In section 1 we generalize the well-known fact that complete Boolean algebras are the injective objects in the category of Boolean algebras and in the category of Heyting algebras, by showing that profinite L-structures are pure-injectives, i.e., have the extension property with respect to the class of pure embeddings.

In section 2 we present a topological characterization of the notion of profinite L-structure, obtaining closure properties of this class of L-structures and a description of its quotients objects. This generalizes results first obtained in [**Lim**] for reduced special groups.

Section 3 introduces an elementary class $\mathcal{A}$ of **L-mod** L-structures, together with a saturated family of congruences, $\mathfrak{C}$, generalizing both the well-known concept of congruence in varieties of algebras and the congruences induced by saturated subgroups of reduced special groups (see [**DM2**]). To each such pair $(\mathcal{A}, \mathfrak{C})$ we associate:

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$$\begin{cases} \text{The profinite hull functor:} & \mathcal{P} : \mathcal{A}^{top} \longrightarrow \mathcal{A}_{pf}; \\ \text{A natural transformation:} & \left(M \xrightarrow{\eta_M} \mathcal{P}(M)\right)_{M \in \mathcal{A}^{top}}, \end{cases}$$

where $\mathcal{A}^{top} \subseteq \mathbf{L-mod^{top}}$ is the (full) subcategory of topological structures in $\mathcal{A}$ and continuous L-morphisms, while $\mathcal{A}_{pf} \subseteq \mathcal{A}^{top}$ is the full subcategory of $\mathcal{A}^{top}$ of profinite structures in $\mathcal{A}$. We show that the profinite hull functor satisfies a universal property, yielding an adjoint pair of functors. This is proven by two different approaches: a categorical one and a topological-analytical one. We also prove that the profinite hull functor preserves directed inductive limits and quotients by a saturated congruence, generalizing results in [**Mrn**].

Section 4 presents constructions and questions suggested by the present work and connected to a local-global principle naturally associated to the profinite hull functor.

1 Profinites and Injectives

For the reader's convenience we register the following

Remark 1 a) Recall that a formula φ in L is

$*$ **positive existential (p.e.)** if it is equivalent to a formula constructed from the atomic formula employing only the connectives $\wedge$, $\vee$ and the existential quantifier $\exists$;

$*$ **positive primitive (p.p.)** if it is equivalent to a formula of the form $\exists\, \overline{x}\, \varphi$, where φ is a conjunction of atomic formulas.

$*$ **geometrical** if it is logically equivalent to one of the form $\forall\, \overline{x}(\varphi(\overline{x}, \overline{y}) \to \psi(\overline{x}, \overline{y}))$, where φ, ψ are p.e.-formulas, **or** to the negation of an atomic formula.

It is well-known that every p.e.-formula is equivalent to the disjunction of finite conjunctions of p.p.-formulas.

b) A map between L-structures, $f : M \longrightarrow N$, is a **pure *L*-morphism** if for each p.e.-formula $\varphi(\vec{x})$ and for all $\overline{a}$ in M, $M \models \varphi[\overline{a}] \Leftrightarrow N \models \varphi[f\overline{a}]$. Hence, a $\underline{L\text{-morphism}}$ $g : M \longrightarrow N$ is pure iff it reflects p.p.-formulas. Clearly, all pure L-morphisms are L-embeddings and any elementary embedding and any L-section [1] are pure L-morphisms. We also mention the following

Fact 2 (See proof of Proposition 3.2.(d) in [**DM3**]) *If Σ is a set of geometrical L-sentences and $f : M \longrightarrow N$ is a pure L-morphism, then $N \models \Sigma \Rightarrow M \models \Sigma$.* □

Proposition 3 *a) Let $M \xrightarrow{f} N \xrightarrow{g} P$ be L-morphisms. Then:*

(1) *f, g pure $\Rightarrow$ $g \circ f$ pure;*

(2) *$g \circ f$ pure $\Rightarrow$ f pure. In particular, every L-section in* **L-mod** *is a pure embedding.*

b) If $f_i : M_i \longrightarrow N_i$, $i \in I$, is a family of pure L-morphisms, their product, $\prod_{i\in I} f_i : \prod_{i \in I} M_i \longrightarrow \prod_{i\in I} N_i$, is a pure L-morphism.

c) Let $\langle I, \leq \rangle$ be an upward directed poset and $\mathcal{M} = \langle M_i;\ \{f_{ij} : i \leq j\} \rangle$ and $\mathcal{N} = \langle N_i;\ \{g_{ij} : i \leq j\} \rangle$ be I-diagrams in **L-mod**. *Let $\underrightarrow{\lim}\, \mathcal{M} = \langle M; f_i \rangle$ and $\underrightarrow{\lim}\, \mathcal{N} = \langle N; g_i \rangle$ their colimits in* **L-mod**. *Let $\langle h_i \rangle_{i \in I}$ $: \mathcal{M} \longrightarrow \mathcal{N}$ be a morphism of I-diagrams and let $\underrightarrow{\lim}\, \overrightarrow{h_i} = h : M \longrightarrow N$ be the limit L-morphism. Then:*

(1) *If each h_i is pure, then $h : M \longrightarrow N$ is pure;*

(2) *If each f_{ij} is pure, $i \leq j$ in I, then $f_i : M_i \longrightarrow M$ is pure.*

d) Let $f : M \longrightarrow N$ be a L-morphism. Then the following conditions are equivalent:

(1) *f is a pure L-embedding;*

(2) *There are a L-structure P, a L-morphism $g : N \longrightarrow P$ and a* pure *L*-embedding *$h : M \longrightarrow P$ such that $g \circ f = h$;*

(3) *There are a L-structure P, a L-morphism $g : N \longrightarrow P$ and a L*-elementary embedding *$h : M \longrightarrow P$ such that $g \circ f = h$;*

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[1] A L-section is a L-morphism that admits a retraction that is also a L-morphism.

(4) *There is an ultrafilter pair* (I, U) *and a* L*-morphism* $g' : N \longrightarrow M^I/U$ *such that* $g' \circ f = \delta_M$*, where* $\delta_M : M \longrightarrow M^I/U$ *is the canonical diagonal* L*-elementary embedding.*

Proof. (Sketch; full proofs appear in [**Mrn**]): Items (a), (b) and (c) are straightforward. For item (d), (1) ⇒(2) is clear: take $P = N$, $g = id$ and $h = f$; (2) ⇒ (1) is item (a.2) above, which also gives (3) ⇒ (1) because elementary embeddings are pure; (1) ⇒ (3) is an application of the Robinson's diagram method; (4) ⇒ (3) is obvious and (3) ⇒ (4) follows from Scott's Lemma (Lemma 8.1.3 in [**BS**]). □

Since any Boolean algebra is the directed union of its (complete) finite subalgebras, it follows from Proposition 3.(c.2) and Sikorsky's Extension Theorem that any *injective* Boolean algebra morphism is a pure embedding in the natural language of Boolean algebras.

As the projective limit of a diagram of complete Boolean algebras and complete homomorphisms is a complete Boolean algebra, it follows that the profinite Boolean algebras are complete, and it is well known that complete Boolean algebras are the injective Boolean (or Heyting) algebras relatively to the class of injective homomorphisms, so profinite Boolean algebras are injectives. We obtain a generalization of this fact: we prove, from the results in [**MM2**] below, that profinite structures are pure injective structures, the structures that have the extension property relatively to the class of pure embeddings.

Remark 4 Recall a non-empty partially ordered set (poset), $\langle I, \leq \rangle$, is *downward directed* if for each $i, j \in I$ there is a $k \in I$ such that $k \leq i, j$. A L-structure is **profinite** if it is L-isomorphic to the limit of a diagram of *finite* L-structures over a *downward directed poset.*

Let $\langle I, \leq \rangle$ be a *downward* directed poset and let $\mathcal{M} = (M_i, \{f_{ij} : i \leq j\})$ be a diagram of finite L-structures over I. Write $(P, \{p_i : i \in I\}) = \underleftarrow{lim}\ \mathcal{M}$. We have the natural L-embedding,

$$\iota : P \hookrightarrow M = \textstyle\prod_{i\in I} M_i,$$

such that for all $i \in I$, $p_i = \pi_i \circ \iota$, where $\pi_i : M \longrightarrow M_i$ is the a canonical projection. It is straightforward that

$$\text{For all } \overline{x} \in P \text{ and all } j,\, k \in I \ (j \in k^{\leftarrow} \ \Rightarrow \ f_{jk}(x_j) = x_k\ ).$$

Moreover, if $\mathcal{F}$ is a filter on I, for each $J \in \mathcal{F}$ there is a natural L-morphism, $\nu_J : M_{|J} = \prod_{j\in J} M_j \longrightarrow M/\mathcal{F}$, given by $x \longmapsto x/\mathcal{F}$, where $M/\mathcal{F}$ is the reduced product $\prod_{i\in I} M_i/\mathcal{F}$. □

In [**MM2**] we prove the following

Theorem 5 *Profinite* L*-structures are retracts of ultraproducts of finite* L*-structures. More precisely, and with the notation in the Remark* 4*, let* $\langle I, \leq \rangle$ *be an downward directed poset and let* $\mathcal{M} = (M_i, \{f_{ij} : i \leq j\})$ *be a diagram of finite* L*-structures over* I*. If* $\underleftarrow{lim}\ \mathcal{M} = (P, \{p_i : i \in I\})$ *then the* L*-morphism given by the composition* $P \xrightarrow{\iota} \prod_{i\in I} M_i \xrightarrow{\nu_I} \prod_{i\in I} M_i/\mathcal{U}$ *is an* L*-section, where* $\mathcal{U}$ *is a* directed ultrafilter *in* I*, i.e.* $i^{\leftarrow} \in \mathcal{U}$*, for each* $i \in I$*.*

$$\begin{tikzcd}
P \arrow[r, "\iota"] \arrow[rrd, "id_P"'] & \prod_{i\in I} M_i \arrow[r, "\nu_I"] & \prod_{i\in I} M_i/\mathcal{U} \arrow[d, "\gamma^U"] \\
 & & P
\end{tikzcd}$$

Corollary 6 *Let* T *be a* L*-theory axiomatized by geometrical* L*-sentences. Then,* Mod(T)*, the full subcategory of models of* T*, is closed under profinite limits* [2]. □

Definition 7 *A* L*-structure* T *is* **pure** $\boldsymbol{L}$**-injective** *if* T *is injective with respect to the class of pure* L*-embeddings, that is, if* M*,* M' *are* L*-structures,* $j : M \rightarrow M'$ *is a pure* L*-embedding and* $g : M \longrightarrow T$ *is a* L*-morphism, there is a* L*-morphism,* $g' : M' \longrightarrow T$*, extending* g*, i.e.,* $g' \circ j = g$*.*

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[2] It is straightforward that the final object of **L-mod** belongs to $\mathcal{A}$.

It is easily established that the in class of pure L-injective structures is closed under products and retracts. We are now establish the main result of this section, namely

Theorem 8 *Profinite L-structures are pure L-injectives.*

Proof. By Theorem 5 profinite L-structures are L-retracts of L-products of finite L-structures; hence, it is enough to prove the following:

Claim: Finite L-structures are pure L-injectives.

Proof of Claim: Let T be a finite L-structure, let $j : M \rightarrow M'$ be a pure L-embedding and let $g : M \longrightarrow T$ be a L-morphism. Proposition 3.(d) yields an ultrafilter pair, (I, U), and a L-morphism $h : M' \longrightarrow M^I/U$ such that $h \circ j = \delta_M$. Now applying the "(I, U)-ultrapower functor" to the L-morphism $g : M \longrightarrow T$ yields $g^I/U \circ \delta_M = \delta_T \circ g$. Since T is *finite*, the diagonal embedding $\delta_T : T \longrightarrow T^I/U$ is a *L-isomorphism.* Now if $g' : M' \longrightarrow T$ is the L-morphism given by the composition $M' \xrightarrow{h} M^I/U \xrightarrow{g^I/U} M'^I/U \xrightarrow{(\delta_T)^{-1}} T$, we obtain $g' \circ j = g$, as needed.

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "g"'] \arrow[dr, "\delta_M"'] & M' \arrow[d, "h"] \\
T \arrow[dr, "\delta_T"'] & M^I/U \arrow[d, "g^I/U"] \\
 & T^I/U
\end{tikzcd}$$

A natural question is if all structures have an "injective hull" relative to some class of embeddings. Since profinite structures are injectives, it is also natural to ask if all structures have a "profinite hull": this is indeed the case, as we shall see in the sections that follow.

2 Profinite Topological Structures

Remark 9 We here recall basic results on congruences and uniform spaces.

a) A *congruence* in a L-structure M is an equivalence relation $C \subseteq M \times M$ which is also a L-substructure of the L-product $M \times M$. If $f : M \longrightarrow N$ is any L-morphism and $C' \subseteq N \times N$ is a congruence in N then $f^\star(C') \doteq (f \times f)^{-1}[C'] \subseteq M \times M$ is a congruence in M, in particular, $ker(f) \doteq f^\star(\Delta_N) = \{(x, x') \in M \times M : f(x) = f(x') \in N\}$ is a congruence in M. As in the universal algebra situation, to each $C \in Cong(M) \doteq \{\text{congruences in } M\}$, there is a canonical L-structure M/C defined on the quotient set: constants and functions symbols are treated as usual in universal algebra; for a k-ary relation symbol R in L, its interpretation in the quotient is defined as follows:

$$(x_0/C, \ldots, x_{k-1}/C) \in R^{M/C} \quad \Leftrightarrow \quad \exists\, x'_0, \ldots, x'_{k-1} \in M \text{ with } x'_i/C = x_i/C,\ i < k \text{ and } (x'_0, \ldots, x'_{k-1}) \in R^M.$$

We also have the fundamental theorem of L-morphisms:

Proposition 10 (cf. Proposition 17.21, p. 174, [**Mir**]) *If M is a L-structure and $C \in Cong(M)$, the natural map $q_C : M \longrightarrow M/C$, $x \mapsto x/C$ is a L-morphism. Moreover, if $f : M \longrightarrow N$ is a L-morphism such that $C \subseteq ker(f)$, there is an unique L-morphism, $\bar{f} : M/C \longrightarrow N$ such that $f = \bar{f} \circ q_C$.* □

A L-structure M is said to be ***L*-inhabited** if for each L-relational symbol R, $R^M \neq \emptyset$; in particular, $M \neq \emptyset$ because $\Delta_M = (=)^M \neq \emptyset$. If $\{M_i, i \in I\}$ is a family of L-inhabited structures and F is a filter in I then the reduced product $\prod_{i \in I} M_i/F$ is a quotient of the product structure $\prod_{i \in I} M_i$ by the L-congruence determined by the kernel of the natural map from $\prod_{i \in I} M_i$ to $\prod_{i \in I} M_i/F$.

b) We assume the reader is familiar with the concept of **uniform space**, as for instance in [**Bu**] or [**Bou**]. A well-known result due to Tychonoff guarantees that a topological space is *uniformizable* iff it is completely regular. Associated to any uniformity there is a notion of completeness and any Hausdorff uniform space has an essentially unique *completion*. Any compact Hausdorff space (X, τ) is uniformizable, by an uniquely determined uniform structure: the set of all neighborhoods of the diagonal $\Delta_X \subseteq X \times X$ in the product topology, and it is complete with this uniformity. $\square$

Proposition 11 *Let* $\langle I, \leq \rangle$ *be a downward directed poset and* $\mathcal{M} : I \longrightarrow$ **L-mod** *a diagram of* L*-structures, such that for each* $i \in I$, M_i *is finite. As in Remark* 4, *write* $(P, \{p_i : i \in I\}) = \varprojlim_{i\in I} M_i$ *for the the projective limit of* $\mathcal{M}$, $(M, \{\pi_i : i \in I\}) = \prod_{i\in I} M_i$ *for the* L*-structure product and* $\iota : P \hookrightarrow M$ *for the canonical* L*-embedding. Suppose that for each* $i \in I$, M_i *is a* discrete *topological space and* P *is a topological subspace of* M *(endowed with the product topology). Then:*

a) P *is a* boolean *space (i.e. a Hausdorff, compact space with a basis of clopens); moreover:*

$*$ *The* L*-operations in* P *are* continuous functions*;*

$*$ *the* n*-ary* L*-relations in* P *are closed subsets of the product* P^n, $n \in \omega$.

Moreover, the topology in P *is the coarsest such that* p_i *is a* continuous L-morphism*, for all* $i \in I$.

b) For each $i \in I$, $ker(p_i) = \{(\vec{s}, \vec{t}) \in P \times P : p_i(\vec{s}) = p_i(\vec{t})\} = \bigcup\{(P \cap \prod_{j\neq i} M_j \times \{x_i\})^2 \ : x_i \in M_i\}$ *is a congruence of* P*, of discrete finite index (i.e., the quotient topological space* $P/ker(p_i)$ *is discrete and finite). The set* $S = \{ker(p_i) : i \in I\}$ *is a fundamental system of entourages of the (unique) uniformity compatible with the topology of* P.

c) If $\psi(\vec{x}, \vec{y})$ *is a p.e.(L)-formula with* $length(\vec{x}) = n$, $length(\vec{y}) = m$ *and* $\vec{a}$ *is a finite sequence of elements of* P *with* $length(\vec{a}) = n$ *then* $[\psi(\vec{a})]^M \doteq \{\vec{b} \in P^m : P \models \psi[\vec{a}, \vec{b}]\}$ *is a closed subset of* P^m*. Moreover,* $[\psi(\vec{a})]^M \approx \varprojlim_{i\in I} [\psi((p_i)^m \vec{b})]^{M_i}$ *(as topological spaces).*

Proof. a) Since $P = \bigcap\{E_{ij} \subseteq M : i \leqslant j\}$, where $E_{ij} = \{\vec{a} \in \prod_{i\in I} M_i : f_{ij}(a_i) = a_j\} \subseteq M$ is a closed subset of M (because M is Hausdorff), $P \subseteq M$ is closed and thus Boolean space. Let t a n-ary functional symbol; since $t^P : P^n \longrightarrow P$ is the unique function such that $p_i \circ t^P = t^{M_i} \circ (p_i)^n$, for each $i \in I$, t^P must be continuous. If R is a n-ary relational symbol, then R^M is a closed subset of M^n: since $(\prod_{i\in I} M_i)^n \approx \prod_{i\in I} M_i^n$, then $R^M \approx \prod_{i\in I} R^{M_i}$ and $\prod_{i\in I} R^{M_i}$ is a closed subset of $\prod_{i\in I} M_i^n$. Since P is a closed substructure of M, P^n is closed in M^n and hence $R^P = R^M \cap P^n$ is closed in P^n.

b) Fix $i \in I$; since M_i is finite and discrete the diagonal $\Delta_i \subseteq M_i \times M_i$ is a M_i-congruence of finite index, and so $C_i \doteq ker(p_i) = (p_i \times p_i)^{-1}[\Delta_i]$ is a P-congruence of discrete finite index, because $\overline{p_i} : P/C_i \rightarrowtail M_i$ is a *continuous injection*. For each $\vec{s} \in P$, let $C_i(\vec{s}) = \{\vec{t} \in P : (\vec{s}, \vec{t}) \in C_i\} = P \cap (\prod_{j\neq i} M_j \times \{s_i\})$. Now note that, since $(I, \leqslant)$ is downward directed, for each $I' \subseteq_{fin} I$ there is a $k \in I$ such that $\forall i \in I'(k \leqslant i)$ whence $C_k \subseteq \bigcap\{C_i : i \in I'\}$, wherefrom we conclude that $\{C_i(\vec{s}) : i \in I\}$ is a fundamental system of *open* neighborhoods of $\vec{s}$ in P. Since P is Hausdorff, we have $\{\vec{s}\} = \bigcap\{C_i(\vec{s}) : i \in I\}$; hence, if $\vec{t} \neq \vec{s}$, there is $i \in I$ such that $\vec{t} \notin C_i(\vec{s})$ (equivalently, $(\vec{s}, \vec{t}) \notin C_i$), and so $\Delta_P = \bigcap\{C_i : i \in I\}$.

Now let $W \subseteq P^2$ be an open neighborhood of the diagonal Δ_P; we claim that there is $i \in I$ such that $C_i \subseteq W$. Indeed, otherwise for all $i \in I$, $C_i \setminus W \neq \emptyset$, entailing $\bigcap\{C_i \setminus W : i \in I\} \neq \emptyset$ (an intersection of a downward directed sequence of non-empty closed subsets of the compact space P^2), which in turn yields $\Delta_P \setminus W = \bigcap\{C_i : i \in I\} \setminus W \neq \emptyset$, a contradiction.

c) By a), any L*-term* $t(\vec{x}, \vec{y})$ gives rise to a continuous functions on P. Let ψ' be a positive-existential formula equivalent to ψ. Again, it follows from item a) (and induction on the complexity of ψ') that $[\psi(\vec{a})]^P$ is closed in P^m. Since the "interpretation subset" of *p.e.*-formulas are preserved by L-morphism, we obtain a cofiltered system of (closed) subsets $\{[\psi((p_i)^m(\vec{a})]^{M_i} \subseteq (M_i)^m : i \in I\}$ and a continuous function $[\psi(\vec{a})]^P \longrightarrow \varprojlim_{i\in I} [\psi((p_i)^m(\vec{a})]^{M_i}$ between *boolean spaces* (we have $[\psi(\vec{a})]^P \subseteq P^m$ and $\varprojlim_{i\in I} [\psi((p_i)^m(\vec{a})]^{M_i} \subseteq \varprojlim_{i\in I} (M_i)^m \approx P^m$). It is straightforward that the continuous function above is injective; hence, it suffices to prove surjectiveness to establish it to be a homeomorphism. But $\varprojlim_{i\in I} [\psi((p_i)^m(\vec{a})]^{M_i} \approx \{\vec{b} \in P^m : \prod_{i\in I} M_i \models \psi[\vec{a}, \vec{b}]\}$ and, since $P \hookrightarrow M$ is L-pure (see Theorem 5), we get $\{\vec{b} \in P^m : \prod_{i\in I} M_i \models \psi[\vec{a}, \vec{b}]\} = \{\vec{b} \in P^m : \varprojlim_{i\in I} M_i \models \psi[\vec{a}, \vec{b}]\} = [\psi(\vec{a})]^P$, as needed. $\square$

At this point, it is natural to consider the following:

Definition 12 *Let L be a first-order language with equality.*

a) $*$ $\mathbf{L\text{-}mod}^{top}$ *is the category whose objects are the* topological *L-structures (i.e., the L-structures M with a topology τ such that the interpretations of the function symbols are continuous functions) and whose morphisms are the* continuous *L-morphisms.*

$*$ $\mathbf{L\text{-}mod}^{disc}$ *is the full subcategory of* $\mathbf{L\text{-}mod}^{top}$ *whose objects are the* discrete *L-structures.*

$*$ $\mathbf{L\text{-}mod}^{sep}$ *is the full subcategory of* $\mathbf{L\text{-}mod}^{top}$ *whose objects are the topological L-structures such that the interpretations of the relational symbols are closed subsets (of the appropriate product).*

$*$ $\mathbf{L\text{-}mod}^{scomp}$ *is the full subcategory of* $\mathbf{L\text{-}mod}^{sep}$ *whose objects are the* compact *L-structures.*

$*$ $\mathbf{L\text{-}mod}_{fin}$ *is the full subcategory of* $\mathbf{L\text{-}mod}^{top}$ *whose objects are the* finite *L-structures endowed with the* discrete topology.

$*$ $\mathbf{L\text{-}mod}_{pf}$ *is the full subcategory of* $\mathbf{L\text{-}mod}^{top}$ *whose objects are the* profinite *L-structures that are topological structures when considered with the natural* boolean topology.

If M is a L-structure, note that:

$*$ *If $M \in \mathbf{L\text{-}mod}^{top}$, any L_M-term $t(\overline{h}, v_1, \ldots, v_n)$ yields a continuous function $t^M_{\overline{h}} : M^n \longrightarrow M$;*

$*$ *If $M \in \mathbf{L\text{-}mod}^{sep}$ then M is a* Hausdorff space *(Δ_M is closed in M^2) and any positive quantifier free L_M-formula, $\varphi(\overline{h}, v_1, \ldots, v_n)$, yields a closed subset of M^n, $[\varphi(\overline{h})]^M = \{ \overline{a} \in M^n : M \models \varphi[\overline{h}, \overline{a}] \}$.*

$*$ *If $M \in \mathbf{L\text{-}mod}^{scomp}$, any positive existential L_M-formula , $\varphi(\overline{h}, v_1, \ldots, v_n)$ yields a closed subset of M^n, $[\varphi(\overline{h})]^M = \{ \overline{a} \in M^n : M \models \varphi[\overline{h}, \overline{a}] \}$.*

$* \; \mathbf{L\text{-}mod} \cong \mathbf{L\text{-}mod}^{disc} \hookrightarrow \mathbf{L\text{-}mod}^{sep}$ *and* $\mathbf{L\text{-}mod}_{fin} \hookrightarrow \mathbf{L\text{-}mod}_{pf} \hookrightarrow \mathbf{L\text{-}mod}^{scomp}$

b) Let $M \in \mathbf{L\text{-}mod}^{top}$. A collection S of subsets of $M \times M$ is a **pf-system in *M*** *if :*

$*$ *The elements of S are congruences in M, of discrete finite index (i.e., the quotient topological structure M/S is discrete and finite);*

$*$ *S is a fundamental system of entourages of some uniformity compatible with the topology of M.*

Remark that if a structure M in $\mathbf{L\text{-}mod}^{top}$ has a pf-system, then M is a completely regular space. Moreover,

$*$ *If $\mathcal{V}(M) \doteq \{C \in Cong(M) : M/C \in \mathbf{L\text{-}mod}_{fin}\}$ is a pf-system, then $\mathcal{V}(M)$ is the largest pf-system in M.*

$*$ *If $M \in \mathbf{L\text{-}mod}^{scomp}$, then all pf-systems in M induces the* same uniformity *in M. Observe that if $C \in \mathcal{V}(M)$ then $C = (q_C \times q_C)^{-1}[\Delta_{M/C}] \subseteq M \times M$ is a* clopen *subset of $M \times M$.*

Theorem 13 *For $M \in \mathbf{L\text{-}mod}^{top}$ the following are equivalent:*

(1) *M is profinite, i.e., is a projective limit of a downward directed system of discrete and finite L-structures.*

(2) *$M \in \mathbf{L\text{-}mod}^{sep}$, it has a pf-system and it is a Boolean topological space.*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 11. For (2) $\Rightarrow$ (1), if S is a pf-system in M, then $(S, \subseteq)$ is a downward directed poset. For each $\Sigma \in S$, $M/\Sigma \in \mathbf{L\text{-}mod}_{fin}$, with the quotient structure. Now consider the diagram $(S, \subseteq) \longrightarrow \mathbf{L\text{-}mod}_{fin}$, where $(\Sigma \subseteq \Sigma')$ in S is taken to $q_{\Sigma,\Sigma'} : M/\Sigma \twoheadrightarrow M/\Sigma'$, the unique $\mathbf{L\text{-}mod}_{fin}$-morphism such that $(M \overset{q_{\Sigma'}}{\twoheadrightarrow} M/\Sigma') = (M \overset{q_{\Sigma}}{\twoheadrightarrow} M/\Sigma \overset{q_{\Sigma,\Sigma'}}{\twoheadrightarrow} M/\Sigma')$. This yields a profinite L-structure $P \doteq \underset{\Sigma \in S}{\underleftarrow{lim}} M/\Sigma$ and a *continuous L-morphism*, $\delta_S : M \longrightarrow \underleftarrow{lim}_{\Sigma \in S} M/\Sigma$, given by $m \mapsto (m/\Sigma)_{\Sigma \in S}$.

Fix $\Sigma \in S$ and let $\Sigma^* \doteq ker(p_\Sigma) \in Cong(P)$. Let R be a k-ary L-relational symbol; if $(\vec{m}_0, \ldots, \vec{m}_{k-1})$ is in R^P, then, since $\{\Sigma^* \in Cong(P) : \Sigma \in S\}$ is a pf-system in P (Proposition 11), it follows that $\{\prod_{i<k} \Sigma^*(\vec{m}_i) : \Sigma \in S\}$ is a fundamental system of neighborhoods of $(\vec{m}_0, \ldots, \vec{m}_{k-1}) \in P^k$. Since $p_\Sigma : P \twoheadrightarrow M/\Sigma$ is a L-morphism, $(p_\Sigma(\vec{m}_1), \ldots, p_\Sigma(\vec{m}_k)) \in R^{M/\Sigma}$ and so there are $a_0, \ldots, a_{k-1} \in M$ such that $(a_0, \ldots, a_{k-1}) \in R^M$ and $(q_\Sigma(a_0), \ldots, q_\Sigma(a_{k-1})) = (p_\Sigma(\vec{m}_0), \ldots, p_\Sigma(\vec{m}_{k-1}))$. Because $q_\Sigma = p_\Sigma \circ \delta_S$, we see that $(\delta_S(a_0), \ldots, \delta_S(a_{k-1})) \in \prod_{i<k} \Sigma^*(\vec{m}_i)$. This means that $(\delta_S)^k[R^M]$ is a *dense subset* of R^P. In particular, $\delta_S[M]$ is a dense subset of P, because $(\delta_S)^2[\Delta_M]$ is dense in Δ_P. Now we are ready to establish the following

Claim: *δ_S is a $\mathbf{L\text{-}mod}^{top}$-isomorphism.*

Indeed, since M is Hausdorff and S is a pf-system, if $m \neq m'$ in M, there is $\Sigma \in S$ such that $\Sigma(m) \cap \Sigma(m') = \emptyset$ and so $\Delta_M = \bigcap S$; clearly, $ker(\delta_S) = \bigcap S$, and so δ_S is *injective.* For $k \in \mathbb{N}$, since $(\delta_S)^k : M^k \longrightarrow P^k$ is a continuous injection from a compact space into a Hausdorff space, it is a *homeomorphism* onto its image. If R is a k-ary relation, because R^M and R^P are closed in M^k and P^k, respectively, and $(\delta_S)^k[R^M]$ is dense in R^P, it follows that $(\delta_S)^k[R^M] = R^P$. In particular, δ_S is *surjective*, because $(\delta_S)^2[\Delta_M] = \Delta_P$. It remains to check δ_S is a L-embedding: let R a k-ary relation and let $(m_0, \ldots, m_{k-1}) \in M^k$ be such that $(\delta_S)^k(m_0, \ldots, m_{k-1}) \in R^P$; since $(\delta_S)^k[R^M] = R^P$, there is $(m'_0, \ldots, m'_{k-1})$ in $R^M \subseteq M^k$ such that $(\delta_S)^k(m_0, \ldots, m_{k-1}) = (\delta_S)^k(m'_0, \ldots, m'_{k-1})$ and, because $(\delta_S)^k$ is injective, we obtain $(m_0, \ldots, m_{k-1}) \in R^M$, as needed. $\square$

The proof of Theorem 13 shows that condition (2) may be rewritten as:

$$(2') : M \text{ is in } \mathbf{L\text{-}mod}^{scomp} \text{ and has a pf-system.}$$

Theorem 13 yields closure properties of $\mathbf{L\text{-}mod}_{pf}$ and a characterization of its quotient objects.

Corollary 14 *The subcategory* $\mathbf{L\text{-}mod}_{pf} \subseteq \mathbf{L\text{-}mod}^{top}$ *is closed under:*

a) Closed substructures, *i.e. if* M *is profinite and* $M' \subseteq M$ *is a substructure of* M *that is also a closed subset, then* M'*, endowed with the topology induced by* M*, is profinite.*

b) Isomorphism, products and general projective limits.

Proof. a) Clearly, M' is a Boolean (sub)space, with $M' \in \mathbf{L\text{-}mod}^{sep}$; and if S is a pf-system in M, then $S' \doteq \{C \cap (M' \times M') : C \in S\}$ is a pf-system in M'.

b) If $\{M_i : i \in I\}$ is a family in $\mathbf{L\text{-}mod}_{pf}$, clearly $M \doteq \prod_{i \in I} M_i$ is a Boolean space. If R is a k-ary relational symbol then, since $R^M \approx \prod_{i \in I} R^{M_i} \subseteq \prod_{i \in I} (M_i)^k \approx M^k$, R^M is closed in M^k, and so $M \in \mathbf{L\text{-}mod}^{sep}$. If S_i is a pf-system in M_i, $i \in I$, then

$$S \doteq \{(\textstyle\prod_{i \in I'} C_i) \times (\prod_{i \in I \setminus I'} M_i \times M_i) \subseteq (M \times M) : \text{for some } I' \subseteq_{fin} I \text{ and some } C_i \in S_i, i \in I'\}$$

is a pf-system in M. Now if $\mathcal{M} : \mathcal{D} \longrightarrow \mathbf{L\text{-}mod}_{pf}$ is any diagram based on an arbitrary small category $\mathcal{D}$, the same methods employed in the proof of Proposition 11.(a) will establish $\varprojlim_{i \in Obj(\mathcal{D})} M_i \hookrightarrow \prod_{i \in Obj(\mathcal{D})} M_i$ is a closed substructure, as needed. $\square$

Corollary 15 *For* $M \in \mathbf{L\text{-}mod}_{pf}$ *and* $\Sigma \in Cong(M)$*, the following are equivalent:*

(1) $M/\Sigma \in \mathbf{L\text{-}mod}_{pf}$ *with the quotient* $\mathbf{L\text{-}mod}^{top}$*-structure;*

(2) *There is* $X \subseteq \mathcal{V}(M)$ *such that* $\Sigma = \bigcap X$.

Proof. (1) $\Rightarrow$ (2) : Let S' a pf-system in M/Σ and write $q_\Sigma : M \twoheadrightarrow M/\Sigma$ for the canonical $\mathbf{L\text{-}mod}^{top}$-morphism. If $C' \in$ S then, since $\overline{q_\Sigma} : M/(q_\Sigma)^\star(C') \longrightarrow (M/\Sigma)/C'$ is a continuous bijection (it is an $\mathbf{L\text{-}mod}^{top}$-isomorphism), we get $(q_\Sigma)^\star(C') \in \mathcal{V}(M)$. Since M/Σ is Hausdorff, $\Delta_{M/\Sigma} = \bigcap S'$, and so $\Sigma = (q_\Sigma)^\star(\Delta_{M/\Sigma}) = \bigcap \{(q_\Sigma)^\star(C') : C' \in S'\}$.

(2) $\Rightarrow$ (1) : We first show that $M/\Sigma \in \mathbf{L\text{-}mod}^{scomp}$: clearly, it is a compact space and since $X \subseteq \mathcal{V}(M)$, Σ is an intersection of a family of *clopen* subsets of $M \times M$. Thus Σ is closed in $M \times M$ and then M/Σ is Hausdorff. It is straightforward that M/Σ is a topological L-structure, so to conclude that $M/\Sigma \in \mathbf{L\text{-}mod}^{sep}$ it suffices to prove that all L-relations in M/Σ are closed. If R is a k-ary relation, then R^M is closed in M^k and, M^k being compact and $(M/\Sigma)^k$ being Hausdorff, it follows that $(q_\Sigma)^k$ is a *closed function*, and so $R^{M/\Sigma} = (q_\Sigma)^k[R_M]$ is closed in $(M/\Sigma)^k$. It remains to check that M/Σ has a pf-system. Let $W \subseteq (M/\Sigma \times M/\Sigma)$ be an open neighborhood of the diagonal $(\Delta_{M/\Sigma} \subseteq W)$ and take $V = (q_\Sigma \times q_\Sigma)^{-1}[W]$; then $V \subseteq (M \times M)$ is open and $\Sigma = (q_\Sigma \times q_\Sigma)^{-1}[\Delta_{M/\Sigma}] \subseteq V$. If $K = (M \times M) \setminus V$, then K is closed in M^2.

Claim: *There is* $X' \subseteq_{fin} X$ *such that* $K \cap \bigcap X' = \emptyset$.
Note that the conclusion is equivalent to $\Delta_M \subseteq \bigcap X' \subseteq V$. Indeed, if for all $X' \subseteq_{fin} X$ $K \cap \bigcap X' \neq \emptyset$, then by the compactness of $M \times M$, $K \cap \Sigma = K \cap \bigcap X \neq \emptyset$, which is impossible because $K = M^2 \setminus V \subseteq M^2 \setminus \Sigma$.

Hence, for each open W of $M/\Sigma \times M/\Sigma$, with $\Delta_{M/\Sigma} \subseteq W$, there is $X' \subseteq_{fin} X$ such that $\bigcap X' \subseteq (q_\Sigma \times q_\Sigma)^{-1}[W]$; now, Proposition 10 entails $\{(q_\Sigma \times q_\Sigma)[\bigcap X'] : X' \subseteq_{fin} X\}$ is a pf-system in M/Σ. $\square$

3 The Profinite Hull Functor

Definition 16 *Let* $\mathcal{A}$ *be a* full *subcategory of* $\mathbf{L\text{-}mod}$ *and let*

$$\mathfrak{C} \;=\; \{\mathfrak{C}(M) : \mathfrak{C}(M) \subseteq Cong(M),\ M \in \mathcal{A}\}$$

be a collection of sets of congruences, parametrized by the L-structures in $\mathcal{A}$. We say that $\mathfrak{C}$ is **saturated** *if it satisfies the following conditions:*

[sat 1] : *For all $M \in \mathcal{A}$, each element of $\mathfrak{C}(M)$ is a $\mathcal{A}$-congruence, that is, if $\Sigma \in \mathfrak{C}(M)$, then the quotient structure M/Σ is in $\mathcal{A}$; moreover, Δ_M (the identity congruence) and $M \times M$ are in $\mathfrak{C}(M)$;*

[sat 2] : *For all $M \in \mathcal{A}$, $\mathfrak{C}(M)$ is closed under finite intersections;*

[sat 3] : *$\mathfrak{C}$ is stable under inverse images, i.e., if $f : M \longrightarrow M'$ is a L-morphism in $\mathcal{A}$ and $\Sigma' \in \mathfrak{C}(M')$, then $f^\star(\Sigma') \doteq (f \times f)^{-1}[\Sigma']$ is in $\mathfrak{C}(M)$.*

17 Remarks and Examples. a) With notation as in Definition 16, a saturated family of congruences, $\mathfrak{C}$, induces a contravariant functor from the category $\mathcal{A}$ into the category Ω of downward directed posets and increasing functions, as follows:

$$(M \xrightarrow{f} M') \;\mapsto\; (\mathfrak{C}(M') \xrightarrow{f^\star} \mathfrak{C}(M)),$$

i.e., $(id_{M'})^\star = id_{\mathfrak{C}(M')}$ and if $f' : M' \longrightarrow M''$ is an $\mathcal{A}$-morphism then $(f' \circ f)^\star = f^\star \circ f'^\star$. Moreover, each $\Sigma' \in \mathfrak{C}(M')$ yields a *derived $\mathcal{A}$-morphism*, $f_{\Sigma'} : M/f^\star(\Sigma') \rightarrowtail M'/\Sigma'$, the *unique* $\mathcal{A}$-morphism such that $f_{\Sigma'} \circ q_{f^\star(\Sigma')} = q'_{\Sigma'} \circ f$.

b) If L does not contains relational symbols and $\mathcal{A}$ is a variety of algebras, then $\mathcal{A}$ is equationally axiomatizable (Birkhoff's Theorem) and the full family $\{Cong(M) : M \in \mathcal{A}\}$ is saturated. In fact, it is well-known that in this case, for each $M \in \mathcal{A}$, $Cong(M)$ is closed under arbitrary intersections and directed unions, constituting an algebraic lattice under inclusion.

c) If $\mathcal{A}$ is the category of reduced special groups (RSG), then $\mathcal{A}$ is axiomatizable by geometrical sentences and it follows from results in Chapter 2 of [**DM2**] that the class of *saturated subgroups* of each RSG yields a saturated family of congruences in $\mathcal{A}$, Sat. As in the case of Example (b), for each RSG G, $Sat(G)$ is closed under arbitrary intersections and directed unions, also constituting an algebraic lattice under inclusion: its compact elements are the saturated subgroups that are the set of represented elements of a Pfister form over G.

d) If a saturated family of congruences in $\mathcal{A}$, $\mathfrak{C}$, is closed under arbitrary intersections and directed unions and is an algebraic lattice under inclusion (as is the case of Examples (b) and (c), above), then:

$*$ An $\mathcal{A}$-morphism $f : M \longrightarrow M'$ yields an increasing function

$$f_\star : (\mathfrak{C}(M), \subseteq) \longrightarrow (\mathfrak{C}(M'), \subseteq) : \Sigma \in \mathfrak{C}(M) \mapsto \bigcap\{\Gamma' \in \mathfrak{C}(M') : (f \times f)[\Sigma] \subseteq \Gamma'\}.$$

Moreover, we have the following *adjunction :*

$$\text{for each } \Sigma \in \mathfrak{C}(M) \text{ and each } \Gamma' \in \mathfrak{C}(M'),\quad f_\star(\Sigma) \subseteq \Gamma' \;\Leftrightarrow\; \Sigma \subseteq f^\star(\Gamma').$$

$*$ The map $(M \xrightarrow{f} M') \;\mapsto\; (\mathfrak{C}(M) \xrightarrow{f_\star} \mathfrak{C}(M'))$ yields a *covariant* functor from $\mathcal{A}$ to the category Ω of downward directed posets and increasing functions (i.e. $(id_{M'})_\star = id_{\mathfrak{C}(M')}$ and if $f' : M' \longrightarrow M''$ is an $\mathcal{A}$-morphism then $(f' \circ f)_\star = f'_\star \circ f_\star$). □

Henceforth, assume we have a pair $(\mathcal{A}, \mathfrak{C})$ where:

(i) $\mathcal{A} = \mathrm{Mod}(T)$, where T is a theory axiomatized by geometrical L-sentences (cf. Remark 1). In particular $\mathcal{A}$ is closed in **L-mod** *under profinite limits (Corollary 6);*

(ii) $\mathfrak{C}$ is a saturated family of $\mathcal{A}$-congruences, as in Definition 16.

Write $\mathcal{A}^{top} \subseteq$ **L-mod**^{top} for the full subcategory of topological L-structures in $\mathcal{A}$ and continuous L-morphisms. Analogously, we define the subcategories $\mathcal{A}^{sep} \subseteq$ **L-mod**^{sep}, $\mathcal{A}^{scomp} \subseteq$ **L-mod**^{scomp}, $\mathcal{A}^{disc} \subseteq$ **L-mod**^{disc}, $\mathcal{A}_{fin} \subseteq$ **L-mod**$_{fin}$ and $\mathcal{A}_{pf} \subseteq$ **L-mod**$_{pf}$.

18 The Profinite Hull. For $M \in \mathcal{A}^{top}$, let

$$\mathcal{V}(M) \doteq \{C \in \mathfrak{C}(M) : \text{the quotient } \mathcal{A}^{top}\text{-object } M/C \text{ is in } \mathbf{L\text{-}mod}_{fin}\}.$$

If $\Sigma, \Sigma_1, \Sigma_2 \in \mathcal{V}(M)$ and $\Sigma' \in \mathfrak{C}(M)$ is such that $\Sigma \subseteq \Sigma'$, then we have the canonical $\mathcal{A}^{top}$-arrows

$$\begin{cases} \quad M/\Sigma \twoheadrightarrow M/\Sigma' & \text{given by} \quad m/\Sigma \mapsto m/\Sigma'; \\ M/(\Sigma_1 \cap \Sigma_2) \rightarrowtail M/\Sigma_1 \times M/\Sigma_2 & \text{given by} \quad m/\Sigma_1 \cap \Sigma_2 \mapsto (m/\Sigma_1, m/\Sigma_2) \end{cases}$$

and so $\mathcal{V}(M) \subseteq \mathfrak{C}(M)$ is a filter in $(\mathfrak{C}(M), \subseteq)$. In particular, $(\mathcal{V}(M), \subseteq)$ is a *downward directed poset.* We then obtain the **canonical diagram of M**, $D(M) : (\mathcal{V}(M), \subseteq) \longrightarrow \mathcal{A}_{fin}$, where $(\Sigma \subseteq \Sigma')$ in $\mathcal{V}(M)$ is taken to $q_{\Sigma,\Sigma'} : M/\Sigma \twoheadrightarrow M/\Sigma'$, the unique $\mathcal{A}_{fin}$-morphism such that

$$(M \overset{q_{\Sigma'}}{\twoheadrightarrow} M/\Sigma') = (M \overset{q_{\Sigma}}{\twoheadrightarrow} M/\Sigma \overset{q_{\Sigma,\Sigma'}}{\twoheadrightarrow} M/\Sigma').$$

The limit of this diagram yields the $\mathcal{A}_{pf}$-object **profinite hull of M**, $\mathcal{P}(M) \doteq \underset{\Sigma \in \mathcal{V}(M)}{\underleftarrow{lim}} M/\Sigma$, together with a canonical $\mathcal{A}^{top}$-morphism, $\eta_M : M \longrightarrow \mathcal{P}(M)$, given by $m \mapsto (m/\Sigma)_{\Sigma \in \mathcal{V}(M)}$. In more detail, given $\Sigma \in \mathcal{V}(M)$, we have the "projections on quotients", $q_\Sigma : M \twoheadrightarrow M/\Sigma$, $m \mapsto m/\Sigma$ and the "projections of the limit", $p_\Sigma : \mathcal{P}(M) \twoheadrightarrow M/\Sigma$, $(m_C/C)_{C \in \mathcal{V}(M)} \mapsto m_\Sigma/\Sigma$, yielding a commutative cone over the diagram $D(M)$, $(q_\Sigma : M \twoheadrightarrow M/\Sigma)_{\Sigma \in \mathcal{V}(M)}$; then, η_M is the unique arrow such that $p_\Sigma \circ \eta_M = q_\Sigma$, for each $\Sigma \in \mathcal{V}(M)$. $\square$

Remark 19 The same argument used in the paragraph preceding the statement of the Claim in the proof of Theorem 13 shows that if R is a k-ary relation in L, $(\eta_M)^k[R^M]$ is dense in $R^{\mathcal{P}(M)}$, $k \in \mathbb{N}$. In particular, $\eta_M[M]$ is dense in $\mathcal{P}(M)$, because $(\eta_M)^2[\Delta_M]$ is dense in $\Delta_{\mathcal{P}(M)}$. $\square$

To justify the adjective *canonical* employed above, we shall now show that the associations $M \mapsto D(M)$ and $M \mapsto \mathcal{P}(M)$ are functorial, and that the family $\eta = \{\eta_M : M \in \mathcal{A}^{top}\}$ is a natural transformation from the functor $id_{\mathcal{A}^{top}}$ to the functor $\iota \circ \mathcal{P}$, where $\iota : \mathcal{A}_{pf} \hookrightarrow \mathcal{A}^{top}$ is the inclusion functor. As a preliminary to this discussion, we recall the definition of morphism of diagrams over distinct bases, which should be compared with Definition 29.3, p. 349 of [**Mir**]. In Part 6 of the latter reference the reader will find a discussion of change of base in a general setting.

Definition 20 *Let $I = \langle I, \leq \rangle$ and $L = \langle L, \leq \rangle$ be downward directed posets and let $\mathcal{D}$ be a category. Let $\mathcal{G} = \langle G_i, g_{ji} \rangle$ and $\mathcal{H} = \langle H_l, h_{ml} \rangle$ be diagrams in $\mathcal{D}$ over I and L, respectively. A* **morphism**, *$\alpha : \mathcal{G} \longrightarrow \mathcal{H}$, consists of a pair, $\alpha = \langle \mathfrak{a}, \mathfrak{u} \rangle$, where $\mathfrak{a} : L \longrightarrow I$ is an increasing map and $\mathfrak{u} = \{\mathfrak{u}(l) : l \in L\}$ is a set of $\mathcal{D}$-morphisms, $\mathfrak{u}(l) : G_{\mathfrak{a}(l)} \longrightarrow H_l$, such that for all $l \leq k$ in L, $h_{lk} \circ \mathfrak{u}(l) = \mathfrak{u}(k) \circ g_{\mathfrak{a}(l),\mathfrak{a}(k)}$, i.e., the diagram below right is commutative:*

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\mathfrak{u}"] \arrow[d, dash] \arrow[dr, phantom, "\alpha"] & \mathcal{H} \arrow[d, dash] \\
I & L \arrow[l, "\mathfrak{a}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G_{\mathfrak{a}(l)} \arrow[r, "\mathfrak{u}(l)"] \arrow[d, "g_{\mathfrak{a}(l),\mathfrak{a}(k)}"'] & H_l \arrow[d, "h_{lk}"] \\
G_{\mathfrak{a}(k)} \arrow[r, "\mathfrak{u}(k)"'] & H_k
\end{tikzcd}$$

The identity of $\mathcal{G}$ is the pair $\langle Id_I, \{Id_{G_i} : i \in I\} \rangle$; if $\mathcal{K} = \langle K_p, k_{qp} \rangle$ is a diagram in $\mathcal{D}$ over the downward directed poset $\langle P, \leq \rangle$ and $\beta = \langle \mathfrak{b}, \mathfrak{v} \rangle : \mathcal{H} \longrightarrow \mathcal{K}$ is a morphism, then $\beta \circ \alpha \doteq \langle \mathfrak{a} \circ \mathfrak{b}, \mathfrak{v} \odot \mathfrak{u} \rangle$, where for each $p \in P$, $(\mathfrak{v} \odot \mathfrak{u})(p) = \mathfrak{v}(\mathfrak{b}(p)) \circ \mathfrak{u}(\mathfrak{a}(\mathfrak{b}(p))) : G_{\mathfrak{a}(\mathfrak{b}(p))} \longrightarrow K_p$. It is straightforward that the usual rules for composition are satisfied and we obtain the **category of all downward directed $\mathcal{D}$-diagrams**, *$Diag_\Omega(\mathcal{D})$.*

21 The "canonical diagram" Functor. a) Let $f : M \longrightarrow M'$ be a $\mathcal{A}^{top}$-morphism. By hypothesis, the (increasing) function $f^\star : \mathfrak{C}(M') \longrightarrow \mathfrak{C}(M) : \Sigma' \mapsto (f \times f)^{-1}[\Sigma']$ is well defined. For $\Sigma' \in \mathcal{V}(M') \subseteq \mathfrak{C}(M')$, consider the *derived $\mathcal{A}^{top}$-morphism*, $f_{\Sigma'} : M/f^\star(\Sigma') \rightarrowtail M/\Sigma'$ given by $m/f^\star(\Sigma') \mapsto f(m)/\Sigma'$, the unique $\mathcal{A}^{top}$-morphism such that $f_{\Sigma'} \circ q_{f^\star(\Sigma')} = q'_{\Sigma'} \circ f$. Then, $f^\star(\Sigma') \in \mathfrak{C}(M)$ and, since $f_{\Sigma'}$ is an *injective* continuous function into a finite discrete space, we must have $f^\star(\Sigma') \in \mathcal{V}(M) = \{C \in \mathfrak{C}(M) : M/C$ is finite and discrete$\}$. Hence, we get a map,

$$(M \xrightarrow{f} M') \mapsto ((\mathcal{V}(M'), \subseteq) \xrightarrow{f^\star} (\mathcal{V}(M), \subseteq)),$$

yielding a contravariant functor from $\mathcal{A}^{top}$ to the category Ω of downward direct posets and increasing functions: clearly, $(id_{M'})^\star = id_{\mathcal{V}(M')}$; and if $f' : M' \longrightarrow M''$, then $(f' \circ f)^\star = f^\star \circ f'^\star$.

b) Let $f : M \longrightarrow M'$ be a $\mathcal{A}^{top}$-morphism. We have the canonical diagrams $D(M) : (\mathcal{V}(M), \subseteq) \longrightarrow \mathcal{A}_{fin}$, $D(M') : (\mathcal{V}(M'), \subseteq) \longrightarrow \mathcal{A}_{fin}$ and an increasing function $f^\star : (\mathcal{V}(M'), \subseteq) \longrightarrow (\mathcal{V}(M), \subseteq)$. There is a natural way to relate the "parallel" diagrams $D(M) \circ f^\star, D(M') : (\mathcal{V}(M'), \subseteq) \longrightarrow \mathcal{A}_{fin}$: for each $\Sigma' \in \mathcal{V}(M')$ we have $D(M) \circ f^\star(\Sigma') = M/f^\star(\Sigma')$ and the derived morphism $f_{\Sigma'} : M/f^\star(\Sigma') \rightarrowtail M'/\Sigma'$. Therefore, the family $\Phi(f) \doteq (f_{\Sigma'})_{\Sigma' \in \mathcal{V}(M')}$ is a natural transformation from the diagram $D(M) \circ f^\star$ to the diagram $D(M')$: indeed, if $(\Gamma' \subseteq \Sigma') \in \mathcal{V}(M')$, then clearly $(f^\star(\Gamma') \subseteq f^\star(\Sigma')) \in \mathcal{V}(M)$ and $f_{\Sigma'} \circ q_{f^\star(\Gamma') f^\star(\Sigma')} = q'_{\Gamma'\Sigma'} \circ f_{\Gamma'}$.

In fact, we get a covariant functor Υ, from $\mathcal{A}^{top}$ to the category $Diag_\Omega(\mathcal{A}_{fin})$ (as in Definition 20), given by

$$(M \xrightarrow{f} M') \overset{\Upsilon}{\longmapsto} (D(M) \overset{\langle f^\star, \Phi(f) \rangle}{\longrightarrow} D(M')).$$

For functoriality, note that $(id^\star_M, \Phi(id_M)) = (id_{\mathcal{V}(M)}, id_{D(M)})$ and if $f' : M' \longrightarrow M''$, then $(f' \circ f)^\star = f^\star \circ f'^\star$ and $\Phi(f' \circ f) = \Phi(f') \odot \Phi(f)$ holds because for each $\Sigma'' \in \mathcal{V}(M'')$ the diagram below is commutative:

$$\begin{tikzcd}
M \arrow[rr, "f' \circ f"] \arrow[dr, "f"'] \arrow[ddd, "q_\Sigma"'] & & M'' \arrow[ddd, "q''_{\Sigma''}"] \\
& M' \arrow[ur, "f'"'] \arrow[d, "q'_{\Sigma'}"'] & \\
& M'/\Sigma' \arrow[dr, "f'_{\Sigma'\Sigma''}"] & \\
M/\Sigma \arrow[ur, "f_{\Sigma\Sigma'}"] \arrow[rr, "(f' \circ f)_{\Sigma\Sigma''}"'] & & M''/\Sigma''
\end{tikzcd}$$

Where $\Sigma' \doteq f'^\star(\Sigma'')$ and
$\Sigma \doteq (f' \circ f)^\star(\Sigma'') = f^\star(\Sigma')$

The above construction is schematically described as follows:

$$\begin{tikzcd}
\mathcal{A}^{top} \arrow[rrr, "\Upsilon"] & & & Diag_\Omega(\mathcal{A}_{fin}) \\
M \arrow[dd, "f"'] & & \mathcal{V}(M) \arrow[dr, "D(M)"] & \\
& & \Phi(f) \Downarrow & \mathcal{A}_{fin} \\
M' & & \mathcal{V}(M') \arrow[uu, "f^\star"] \arrow[ur, "D(M')"'] &
\end{tikzcd}$$

22 The Profinite Hull Functor. We saw above that there is a "canonical diagram" functor,

$$\Upsilon : \mathcal{A}^{top} \longrightarrow Diag_\Omega(\mathcal{A}_{fin}), \ \text{ given by } \ (M \xrightarrow{f} M') \overset{\Upsilon}{\mapsto} (D(M) \overset{(f^\star, \Phi(f))}{\longrightarrow} D(M')).$$

Furthermore, each $M \in \mathcal{A}^{top}$ has a "profinite hull", $\mathcal{P}(M) \doteq \underleftarrow{lim}\ D(M) \in \mathcal{A}_{pf}$. To show that object-map $M \mapsto \mathcal{P}(M)$ extends to a functor, $\mathcal{P} : \mathcal{A}^{top} \longrightarrow \mathcal{A}_{pf}$, it suffices to prove there is a well-defined *functor limit*, $\underleftarrow{lim}$: $Diag_\Omega(\mathcal{A}_{fin}) \longrightarrow \mathcal{A}_{pf}$. The **profinite hull functor $\mathcal{P}$** will then be the composition of the functors "limit" and "canonical diagram". With notation as in Definition 20, the existence of the "functor limit", $\underleftarrow{lim}$: $Diag_\Omega(\mathcal{A}_{fin}) \longrightarrow \mathcal{A}_{pf}$, is guaranteed by the following general

Fact 23 *Let $\langle I, \leq \rangle$, $\langle L, \leq \rangle$ and $\langle P, \leq \rangle$ be downward directed posets. Let $\mathcal{G} = \langle G_i;\ g_{ji} \rangle$, $\mathcal{H} = \langle H_l;\ h_{ml} \rangle$ and $\mathcal{K} = \langle K_p;\ k_{pq} \rangle$ be diagrams in $\mathcal{A}_{fin}$ over I, L and P, respectively. Let $\alpha = \langle \mathfrak{a}, \mathfrak{u} \rangle : \mathcal{G} \longrightarrow \mathcal{H}$ and $\beta : \mathcal{H} \longrightarrow \mathcal{K}$ be change of base morphisms. Let $\langle \widehat{G}, g_i \rangle = \underleftarrow{lim}\ \mathcal{G}$, $\langle \widehat{H}, h_l \rangle = \underleftarrow{lim}\ \mathcal{H}$ and $\langle \widehat{K}, k_p \rangle = \underleftarrow{lim}\ \mathcal{K}$ be the corresponding projective limits. Then:*

(1) *The family $(\widehat{G} \xrightarrow{g_{\mathfrak{a}(l)}} G_{\mathfrak{a}(l)} \xrightarrow{\mathfrak{u}(l)} H_l)_{l \in L}$ is a* commutative cone *over the diagram $\mathcal{H}$. Hence, there is an* unique *$\mathcal{A}_{pf}$-morphism, $\widehat{\alpha} : \widehat{G} \longrightarrow \widehat{H}$, such that for each $l \in L$, $h_l \circ \widehat{\alpha} = \mathfrak{u}(l) \circ g_{\mathfrak{a}(l)}$.*

(2) *$\widehat{(\beta \circ \alpha)} = \widehat{\beta} \circ \widehat{\alpha}$ and if $\alpha = id_{\mathcal{G}}$ then $\widehat{\alpha} = id_{\widehat{G}}$.* ☐

Hence, the **Profinite Hull Functor**, $\mathcal{P} : \mathcal{A}^{top} \longrightarrow \mathcal{A}_{pf}$, is given by:

<u>Objects</u> : $M \in \mathcal{A}^{top} \mapsto \mathcal{P}(M) \doteq (\underleftarrow{lim}_{\Sigma \in \mathcal{V}(M)} M/\Sigma) \in \mathcal{A}_{pf}$ and $(\mathcal{P}(M) \xrightarrow{p_\Sigma} M/\Sigma)_{\Sigma \in \mathcal{V}(M)}$ is the limit cone.

<u>Morphisms</u> : $(M \xrightarrow{f} M') \in \mathcal{A}^{top} \ \mapsto \ (\mathcal{P}(M) \xrightarrow{\mathcal{P}(f)} \mathcal{P}(M')) \in \mathcal{A}_{pf}$, where $\mathcal{P}(f) : \mathcal{P}(M) \longrightarrow \mathcal{P}(M')$ is the

unique $\mathcal{A}_{pf}$-morphism such that for each $\Sigma' \in \mathcal{V}(M')$, $p'_{\Sigma'} \circ \mathcal{P}(f) = f_{\Sigma'} \circ p_{f^\star(\Sigma')}$. $\square$

24 The natural transformation η. The family $(M \xrightarrow{\eta_M} \mathcal{P}(M))_{M \in \mathcal{A}^{top}}$ is a natural transformation from the functor $id_{\mathcal{A}^{top}}$ to the functor $\iota \circ \mathcal{P}$, where $\iota : \mathcal{A}_{pf} \hookrightarrow \mathcal{A}^{top}$ is the inclusion functor. It suffices to check that if $f : M \longrightarrow M'$ is a morphism in $\mathcal{A}^{top}$, then $\mathcal{P}(f) \circ \eta_M = \eta_{M'} \circ f$. Equivalently, by the universal property of $\mathcal{P}(M') = \underleftarrow{lim}_{\Sigma' \in \mathcal{V}(M')} M'/\Sigma'$, it must be verified that for each $\Sigma' \in \mathcal{V}(M')$, $p'_{\Sigma'} \circ \mathcal{P}(f) \circ \eta_M = p'_{\Sigma'} \circ \eta_{M'} \circ f$. But this follows directly from the definitions and a straightforward diagram chase:

$$\begin{tikzcd}
M \arrow[rr, "q_{f^\star(\Sigma')}"] \arrow[dr, "\eta_M"'] \arrow[ddd, "f"'] & & M/f^\star(\Sigma) \arrow[ddd, "f_{\Sigma'}"] \\
& \mathcal{P}(M) \arrow[ur, "p_{f^\star(\Sigma')}"'] \arrow[d, "\mathcal{P}(f)"'] & \\
& \mathcal{P}(M') \arrow[dr, "p'_{\Sigma'}"] & \\
M' \arrow[ur, "\eta_{M'}"] \arrow[rr, "q'_{\Sigma'}"'] & & M'/\Sigma'
\end{tikzcd}$$

$$\begin{aligned} p'_{\Sigma'} \circ \mathcal{P}(f) \circ \eta_M &= f_{\Sigma'} \circ p_{f^\star(\Sigma')} \circ \eta_M \\ &= f_{\Sigma'} \circ q_{f^\star(\Sigma')} \\ &= q'_{\Sigma'} \circ f \\ &= p'_{\Sigma'} \circ \eta_{M'} \circ f \end{aligned}$$

In [**Mrn**] it is shown that:

$*$ For each Boolean algebra B, the BA-morphism $\eta_B : B \longrightarrow \mathcal{P}(B)$ may be identified with the natural BA-morphism injective $B \rightarrowtail Parts(Stone(B)) : b \mapsto \{U \in Stone(B) : b \in U\}$;

$*$ For each reduced special group G, the RSG-morphism $\eta_G : G \longrightarrow \mathcal{P}(G)$ reflects subforms; in particular, it is a complete embedding and reflects isotropy of forms over G.

We now show that the functor $\mathcal{P} : \mathcal{A}^{top} \longrightarrow \mathcal{A}_{pf}$ is a profinite hull: for each M in $\mathcal{A}^{top}$, every morphism from M to an object in $\mathcal{A}_{pf}$ factors uniquely through the natural arrow $\eta_M : M \longrightarrow \mathcal{P}(M)$; and then prove that $\mathcal{P}$ preserves inductive limits and quotients.

25 A universal property. Let $f : M \longrightarrow P$ be a $\mathcal{A}^{top}$-morphism, where $M \in \mathcal{A}^{top}$ and $P \in \mathcal{A}_{pf}$. Since $\mathcal{V}(P)$ is a pf-system in P, $\eta_P : P \longrightarrow \mathcal{P}(P)$ is an $\mathcal{A}_{pf}$-isomorphism (see the proof of Theorem 13). Hence, we may obtain an extension $\widetilde{f} \in \mathcal{A}_{pf}(\mathcal{P}(M), P)$ of f to $\mathcal{P}(M)$ (i.e. $f = \widetilde{f} \circ \eta_M$), given by $\widetilde{f} \doteq (\eta_P)^{-1} \circ \mathcal{P}(f)$. Moreover, this continuous extension is unique, since $\eta_M[M]$ is dense in $\mathcal{P}$(M) (cf. Remark 19) and P is a Hausdorff space. Therefore, each $\mathcal{A}^{top}$-morphism, $f : M \longrightarrow P$, with $M \in \mathcal{A}^{top}$and $P \in \mathcal{A}_{pf}$, has a unique extension, $\widetilde{f} \in \mathcal{A}_{pf}(\mathcal{P}(M), P)$, such that $f = \widetilde{f} \circ \eta_M$. In particular, the functor $\mathcal{P} : \mathcal{A}^{top} \longrightarrow \mathcal{A}_{pf}$ is left adjoint to the inclusion functor $\iota : \mathcal{A}_{pf} \hookrightarrow \mathcal{A}^{top}$ and the natural transformation η is the unit of this adjunction. $\square$

The Theorem that follows uses a topological-analytical technique to obtain a generalization of the above result, also providing a characterization of the universal arrow η_M.

Theorem 26 *Let $M \in \mathcal{A}^{top}$.*

a) Let $K \in \mathcal{A}^{sep}$ that has a pf-system with respect to which it is a complete Hausdorff uniform space. Then for each $f \in \mathcal{A}^{top}(M, K)$ such that $ker(f) \supseteq ker(\eta_M) = \bigcap\{ker(p_\Sigma) : \Sigma \in \mathcal{V}(M)\}$, there is an unique $\mathcal{A}^{top}$-morphism, $\widetilde{f} : \mathcal{P}(M) \longrightarrow K$, such that $\widetilde{f} \circ \eta_M = f$. In particular, these conditions are satisfied if $K \in \mathcal{A}_{pf}$ and thus each $f \in \mathcal{A}^{top}(M, K)$ has an unique extension $\widetilde{f}$ to $\mathcal{P}(M)$, satisfying $\widetilde{f} = \eta_K^{-1} \circ \mathcal{P}(f)$.

b) Let $T \in \mathcal{A}_{pf}$, let $j : M \longrightarrow T$ be an $\mathcal{A}^{top}$-morphism and let $M^+ \doteq j[M] \subseteq T$ be the image structure. Assume that:

$*$ $ker(j) \subseteq ker(\eta_M)$ (hence, η_M factors uniquely through M^+, i.e. $M^+ \cong M/ker(j)$ and $\eta_M = (\eta_M)^+ \circ j$);

$*$ The L-morphism $(\eta_M)^+ : M^+ \longrightarrow \mathcal{P}(M)$ is uniformly continuous in the uniformity induced by T on M^+ (i.e., for each $\Sigma \in \mathcal{V}(M)$, there is $C \in \mathcal{V}(T)$ such that $(\eta_M^+ \times \eta_M^+)[C \cap (M^+ \times M^+)] \subseteq ker(p_\Sigma)$).

$*$ The image of each L-relation in M (including equality) is dense in the corresponding L-relation in T in the product topology (i.e., if R is a n-ary relation, $C \in \mathcal{V}(T)$ and $(x_0, \ldots, x_{n-1}) \in R^T \subseteq T^n$, there are $a_0, \ldots, a_{n-1}$ in M such that $(a_0, \ldots, a_{n-1}) \in R^M$ and $(j(a_i), x_i) \in C$).

Then, with the notation as in (a), $\widetilde{j} : \mathcal{P}(M) \longrightarrow T$ *is the* unique $\mathcal{A}_{pf}$-isomorphism *such that* $\widetilde{j} \circ \eta_M = j$. *In particular,* η_M *satisfies above conditions.*

Proof. a) As $\eta_M[M]$ is a dense subset of $\mathcal{P}(M)$ and K is an Hausdorff space there is *at most one* continuous "extension" of f to $\mathcal{P}(M)$. We will construct an $\mathcal{A}^{top}$-extension f to $\mathcal{P}(M)$. Let $M' = \eta_M[M]$ be the image L-structure by η_M (so $M' \cong M/ker(\eta_M)$ in **L-mod**), endowed with the uniformity induced by $\mathcal{P}(M)$. By Proposition 10, there is an unique L-morphism, $f' : M' \longrightarrow K$, such that $f' \circ (\eta_M)_{\upharpoonright} = f$. We will show that f' is uniformly continuous for some fixed pf-system S in K as in the hypothesis and let $S' \doteq \{ker(p_\Sigma) \cap (M' \times M') : \Sigma \in \mathcal{V}(M)\}$ be a fundamental system of entourages of M'. Given $C \in S$, set $\Sigma \doteq (f \times f)^{-1}[C]$ and $C' \doteq ker(p_\Sigma) \cap (M' \times M')$; then $\Sigma \in \mathcal{V}(M)$ and

$$C' \ = \ \{(a', b') \in M' \times M' : \exists x, y \in M \ (a', b') = (\eta_M(x), \eta_M(y)) \text{ and } p_\Sigma(a') = p_\Sigma(b')\},$$

hence, $(f' \times f')[C'] \ = \ \{(f(x), f(y)) : x, y \in M \ , \ q_\Sigma(x) = q_\Sigma(y)\} \ = \ (f \times f)[\Sigma] \subseteq C$, showing that f' is indeed uniformly continuous. Since M' is dense in $\mathcal{P}(M)$ and both $\mathcal{P}(M)$ and K are complete uniform spaces, there is an *unique* uniformly continuous map, $\widetilde{f} : \mathcal{P}(M) \longrightarrow K$ such that $\widetilde{f} \circ \iota = f'$, where $\iota : M' \hookrightarrow \mathcal{P}(M)$. Hence, $\widetilde{f} : \mathcal{P}(M) \longrightarrow K$ is a *continuous function*, satisfying $\widetilde{f} \ \circ \ \eta_M = \widetilde{f} \ \circ \ \iota \ \circ \ (\eta_M)_{\upharpoonright} = f' \ \circ \ (\eta_M)_{\upharpoonright} = f$, and $\widetilde{f}$ is the *unique* continuous extension of f along η_M, as needed.

$$\begin{tikzcd}
M \arrow[rr, "\eta_M"] \arrow[dr, "\eta_{M\upharpoonright}"] \arrow[ddr, "f"'] & & \mathcal{P}(M) \arrow[ddl, "\widetilde{f}"] \\
& M' \arrow[ur, "\iota"'] \arrow[d, "f'"] & \\
& K &
\end{tikzcd}$$

It remains to check that $\widetilde{f}$ is a L-morphism. Note that M' is a L-subalgebra of $\mathcal{P}(M)$, i.e. it contains the interpretations in $\mathcal{P}(M)$ of the constants in L, and is closed under the interpretations of all L-operations in $\mathcal{P}(M)$. If c is a constant in L, then since f' is a L-morphism, we get $c^K = f'(c^{M'}) = \widetilde{f}(c^{\mathcal{P}(M)})$, as needed. If t is a n-ary operation in L, let $h_1, h_2 : (\mathcal{P}(M))^n \longrightarrow K$ be the continuous maps

$$((\mathcal{P}(M))^n \xrightarrow{(\widetilde{f})^n} K^n \xrightarrow{t^K} K) \quad \text{and} \quad ((\mathcal{P}(M))^n \xrightarrow{t^{\mathcal{P}(M)}} \mathcal{P}(M) \xrightarrow{\widetilde{f}} K),$$

respectively and set $h : (\mathcal{P}(M))^n \longrightarrow K \times K : (\vec{m}_0, \ldots, \vec{m}_{n-1}) \mapsto (h_1(\vec{m}_0, \ldots, \vec{m}_{n-1}), h_2(\vec{m}_0, \ldots, \vec{m}_{n-1}))$. Since the diagonal Δ_K is closed in K^2 (K is Hausdorff)and h is a continuous map it follows that $h^{-1}[\Delta_K]$ is closed in $(\mathcal{P}(M))^n$. Since M' is a L-subalgebra of $\mathcal{P}(M)$, $\widetilde{f} \circ \iota = f'$ and f' is an L-morphism, we obtain

$$\begin{aligned}
(M')^n \ &= \ \{(\vec{m}_0, \ldots, \vec{m}_{n-1}) \in (M')^n : f'(t^{M'}((\vec{m}_0, \ldots, \vec{m}_{n-1})) = t^K((f')^n(\vec{m}_0, \ldots, \vec{m}_{n-1})\} \\
&\subseteq \ \{(\vec{m}_0, \ldots, \vec{m}_{n-1}) \in (\mathcal{P}(M))^n : \widetilde{f}(t^{\mathcal{P}(M)}((\vec{m}_0, \ldots, \vec{m}_{n-1})) = t^K((\widetilde{f})^n(\vec{m}_0, \ldots, \vec{m}_{n-1})\} \ = \ h^{-1}[\Delta_K].
\end{aligned}$$

Since M' is dense in $\mathcal{P}(M)$, the same is true of $(M')^n$ in $(\mathcal{P}(M))^n$, and so $h^{-1}[\Delta_K]$ is a dense closed subset of $(\mathcal{P}(M))^n$. Thus, $h^{-1}[\Delta_K] = (\mathcal{P}(M))^n$, i.e. $\widetilde{f}$ preserves the operation t. Let R an n-ary L-relation and let $(\vec{m}_0, \ldots, \vec{m}_{n-1}) \in R^{\mathcal{P}(M)} \subseteq (\mathcal{P}(M))^n$. Since R^K is closed in K^n, to prove $(\widetilde{f})^n(\vec{m}_0, \ldots, \vec{m}_{n-1}) \in R^K$ it suffices to show that for each neighborhood W of $(\widetilde{f})^n(\vec{m}_0, \ldots, \vec{m}_{n-1}) \in K^n$ we have $W \cap R^K \neq \emptyset$. Since $\widetilde{f}$ is continuous and $\{\prod_{i<n}(ker(p_\Sigma))(\vec{m}_i) : \Sigma \in \mathcal{V}(M)\}$ is a fundamental system of neighborhoods of $(\vec{m}_0, \ldots, \vec{m}_{n-1}) \in (\mathcal{P}(M))^n$, there is $\Sigma \in \mathcal{V}(M)$ such that $(\widetilde{f})^n[\prod_{i<n}(ker(p_\Sigma))(\vec{m}_i)] \subseteq W$. Since $(\eta_M)^n[R^M]$ is dense in $R^{\mathcal{P}(M)}$ (see Remark 19), there is $(a_0, \ldots, a_{n-1}) \in R^M$ such that $(\eta_M)^n(a_0, \ldots, a_{n-1}) \in R^{\mathcal{P}(M)} \cap \prod_{i<n}(ker(p_\Sigma))(\vec{m}_i)$; from $\widetilde{f} \circ \eta_M = f$ we get $(\widetilde{f})^n((\eta_M)^n(a_0, \ldots, a_{n-1})) = f^n(a_0, \ldots, a_{n-1}) \in R^K$, showing that $W \cap R^K \neq \emptyset$, completing the proof that $\widetilde{f}$ is a L-morphism.

It is clear that if K is in $\mathcal{A}_{pf}$, then it satisfies the properties in statement of (a) and, since η is a natural transformation and $\eta_K : K \xrightarrow{\cong} \mathcal{P}(K)$, we have $ker(\eta_M) \subseteq ker(\mathcal{P}(f) \ \circ \ \eta_M) \ = \ ker(\eta_K \ \circ \ f) \ = \ ker(f)$, as well as $(\eta_K)^{-1} \ \circ \ \mathcal{P}(f) \ = \ \widetilde{f}$, for each $f : M \longrightarrow K$, as needed.

b) The conditions on T and $j : M \longrightarrow T$ are such that the proof of item (a) yields a *unique* $\mathcal{A}_{pf}$-morphism $\widehat{\eta}_M : T \longrightarrow \mathcal{P}(M)$, such that $\widehat{\eta}_M \circ j = \eta_M$. Then, $\widehat{\eta}_M \circ \widetilde{j} \circ \eta_M = \eta_M$ and it follows from (a) that $\widehat{\eta}_M \circ \widetilde{j} = id_{\mathcal{P}(G)}$.

Since $\widetilde{j} \circ \widehat{\eta}_M \circ j = j$, $j[G]$ is dense in T and T is a Hausdorff, we obtain $\widetilde{j} \circ \widehat{\eta}_M = id_T$. Hence, $\widetilde{j}$ and $\widehat{\eta}_M$ are the unique (inverse) $\mathcal{A}_{pf}$-isomorphism between the arrows $\eta_M : M \longrightarrow \mathcal{P}(M)$ and $j : M \longrightarrow T$. $\square$

A well-known general categorial result on adjoint pairs of functors yields

Corollary 27 *The inclusion functor* $\iota : \mathcal{A}_{pf} \hookrightarrow \mathcal{A}^{top}$ *preserves projective limits and* $\mathcal{P} : \mathcal{A}^{top} \longrightarrow \mathcal{A}_{pf}$ *preserves inductive limits. In particular, since* $\mathcal{A}$ *is a* $\forall\exists$*-axiomatizable elementary class,* $\mathcal{A}^{top} \hookrightarrow \mathbf{L\text{-}mod}^{top}$ creates *upward directed limits, i.e. if* $(I, \leqslant)$ *is an upward directed poset and* $D : (I, \leqslant) \longrightarrow \mathcal{A}^{top}$ *is a diagram, then the inductive limit in the category* $\mathcal{A}^{top}$ *of the composition* $(I, \leqslant) \xrightarrow{D} \mathcal{A}^{top} \hookrightarrow \mathbf{L\text{-}mod}^{top}$, $(M, (D(i) \xrightarrow{\alpha_i} M)_{i \in I})$, *is also the inductive limit of the diagram* D *in the category* $\mathcal{A}^{top}$, *thus* $(\mathcal{P}(M), (\mathcal{P}(D(i)) \xrightarrow{\mathcal{P}(\alpha_i)} \mathcal{P}(M))_{i \in I})$ *is the inductive limit in the category* $\mathcal{A}_{pf}$ *of the diagram* $\mathcal{P} \circ D : (I, \leqslant) \longrightarrow \mathcal{A}_{pf}$. $\square$

Before stating the pertinent result for quotients we register the following

Remark 28 Let $f : M \longrightarrow N$ be a $\mathcal{A}^{top}$-morphism with *dense image.* Since η_N is also a $\mathcal{A}^{top}$-morphism with dense image and $\mathcal{P}(f) \circ \eta_M = \eta_N \circ f$, we conclude $\mathcal{P}(f) : \mathcal{P}(M) \longrightarrow \mathcal{P}(N)$ has dense image; since it is continuous and the spaces involved are compact Hausdorff, $\mathcal{P}(f)$ is a closed surjective map. Hence:

(1) $ker(\mathcal{P}(f))$ is a *closed* congruence in $\mathfrak{C}(\mathcal{P}(M))$ and $\mathcal{P}(M)/ker(\mathcal{P}(f))$ is a Boolean space;

(2) The derived $\mathcal{A}^{top}$-arrow from $\mathcal{P}(f)$ (via Proposition 10), $g : \mathcal{P}(M)/ker(\mathcal{P}(f)) \longrightarrow \mathcal{P}(N)$, is a bijective $\mathcal{A}^{top}$-morphism and a homeomorphism of Boolean spaces.

A natural question is to know when is g^{-1} a $\mathcal{A}^{top}$*-morphism.* Note that if g is an $\mathcal{A}^{top}$*-isomorphism* then, by Corollary 15, $ker(\mathcal{P}(f))$ is the intersection of some subfamily of $\mathcal{V}(\mathcal{P}(M))$; in the course of the proof of Theorem 29 below we shall show that for a congruence Θ in $\mathfrak{C}$, this condition is also sufficient. $\square$

Regarding quotients, we now state

Theorem 29 *The functor* $\mathcal{P} : \mathcal{A}^{top} \longrightarrow \mathcal{A}_{pf}$ *preserves quotients. More precisely, for* $M \in \mathcal{A}^{top}$ *and* $\Theta \in \mathfrak{C}(M)$, *let* $q_\Theta : M \twoheadrightarrow M/\Theta$ *be the quotient* $\mathcal{A}^{top}$*-morphism and let* $\Sigma_\Theta = ker(\mathcal{P}(q_\Theta)) \in \mathfrak{C}(\mathcal{P}(M))$. *Then:*

a) $\mathcal{P}(q_\Theta) : \mathcal{P}(M) \longrightarrow \mathcal{P}(M/\Theta)$ *is a surjective* $\mathcal{A}_{pf}$*-morphism.*

b) If $g_\Theta : \mathcal{P}(M)/\Sigma_\Theta \longrightarrow \mathcal{P}(M/\Theta)$ *is the derived* $\mathcal{A}^{top}$*-morphism from* $\mathcal{P}(q_\Theta)$ *(via* Proposition 10*),* g_Θ *is a* $\mathcal{A}_{pf}$*-isomorphism.*

Proof. Item (a) follows immediately from Remark 28.

$$\begin{tikzcd}
 & \mathcal{P}(M) \arrow[d, "\mathcal{P}(q_\Theta)"'] \arrow[dr, "q_{\Sigma_\Theta}"] & \\
M \arrow[ur, "\eta_M"] \arrow[dr, "q_\Theta"'] & \mathcal{P}(M/\Theta) \arrow[r, shift right, "h_\Theta"'] & \mathcal{P}(M)/\Sigma_\Theta \arrow[l, shift right, "g_\Theta"'] \\
 & M/\Theta \arrow[u, "\eta_{M/\Theta}"] \arrow[ur, "(\eta_M)_{\Theta,\Sigma_\Theta}"'] &
\end{tikzcd}$$

Fact. *With notation as above,* Σ_Θ *is the intersection of a subfamily of* $\mathcal{V}(\mathcal{P}(M))$.

Proof. Consider the subposet $\mathcal{V}_\Theta(M) = \{\Omega \in \mathcal{V}(M) : \Theta \subseteq \Omega\}$; then, Proposition 10 entails yields:

$*$ A pair of inverse bijective increasing functions

$$\begin{cases} (i)\ (q_\Theta)^\star \restriction : \mathcal{V}(M/\Theta) \xrightarrow{\cong} \mathcal{V}_\Theta(M) \text{ given by } \Gamma \in \mathcal{V}(G/\Theta) \mapsto (q_\Theta \times q_\Theta)^{-1}[\Gamma] \in \mathcal{V}_\Theta(G); \\ (ii)\ (q_\Theta)_\star \restriction : \mathcal{V}_\Theta(M) \xrightarrow{\cong} \mathcal{V}(M/\Theta), \text{ given by } \Omega \in \mathcal{V}_\Theta(M) \mapsto (q_\Theta \times q_\Theta)[\Omega] \ = \ \Omega/\Theta \ \in \ \mathcal{V}_\Theta(M). \end{cases}$$

$*$ A canonical $\mathcal{A}_{pf}$-isomorphism, $\alpha_\Theta \ : \underset{\longleftarrow \Gamma \in \mathcal{V}(M/\Theta)}{lim} (M/\Theta)/\Gamma \xrightarrow{\cong} \underset{\longleftarrow \Omega \in \mathcal{V}_\Theta(M)}{lim} M/\Omega.$

Now, the definition of the functor $\mathcal{P}$ guarantees that $\mathcal{P}(q_\Theta) : \varprojlim_{\Omega\in\mathcal{V}(M)} M/\Omega \twoheadrightarrow \varprojlim_{\Gamma\in\mathcal{V}(M/\Theta)} (M/\Theta)/\Gamma$ is the surjective $\mathcal{A}_{pf}$-morphism such that $(m_\Omega/\Omega)_{\Omega\in\mathcal{V}(M)} \in \mathcal{P}(M) \mapsto (\,(m_\Omega/\Theta)\ /\ (\Omega/\Theta)\,)_{\Omega/\Theta\in\mathcal{V}(M/\Theta)} \in \mathcal{P}(M/\Theta)$. It is straightforward that the composition $\varprojlim_{\Omega\in\mathcal{V}(M)} M/\Omega \xrightarrow{\mathcal{P}(q_\Theta)} \varprojlim_{\Gamma\in\mathcal{V}(M/\Theta)} (M/\Theta)/\Gamma \xrightarrow{\alpha_\Theta} \varprojlim_{\Omega\in\mathcal{V}_\Theta(M)} M/\Omega$ is the "projection" $\mathcal{A}_{pf}$-morphism

$$\rho_\Theta : \varprojlim_{\Omega\in\mathcal{V}(M)} M/\Omega \longrightarrow \varprojlim_{\Omega\in\mathcal{V}_\Theta(M)} M/\Omega, \quad \text{given by} \quad (m_\Omega/\Omega)_{\Omega\in\mathcal{V}(M)} \mapsto (m_\Omega/\Omega)_{\Omega\in\mathcal{V}_\Theta(M)}.$$

Hence, $\Sigma_\Theta \doteq ker(\mathcal{P}(q_\Theta)) = ker(\mathcal{P}(q_\Theta)\circ\alpha_\Theta) = ker(\rho_\Theta) = \bigcap\{ker(p_\Omega) : \Omega\in\mathcal{V}_\Theta(M)\}$. But if $\Omega\in\mathcal{V}(M)$, then $\overline{p}_\Omega : \mathcal{P}(M)/ker(p_\Omega) \longrightarrow M/\Omega$ is a $\mathcal{A}_{fin}$-isomorphism, whence $ker(p_\Omega) \in \mathcal{V}(\mathcal{P}(M))$, showing that Σ_Θ is the intersection of a subfamily of $\mathcal{V}(\mathcal{P}(M))$, as needed.

Since $\eta_{M/\Theta}\circ q_\Theta = \mathcal{P}(q_\Theta)\circ\eta_M$, it follows that $\Theta \subseteq \eta_M^*(\Sigma_\Theta)$ and so there is a unique $\mathcal{A}^{top}$-morphism, $(\eta_M)_{\Theta,\Sigma_\Theta} : M/\Theta \longrightarrow \mathcal{P}(M)/\Sigma_\Theta$, such that $(\eta_M)_{\Theta,\Sigma_\Theta}\circ q_\Theta = q_{\Sigma_\Theta}\circ\eta_M$. By the Fact above, Σ_Θ is the intersection of some subfamily of $\mathcal{V}(\mathcal{P}(M))$ and so, by Corollary 15, $\mathcal{P}(M)/\Sigma_\Theta \in \mathcal{A}_{pf}$. Now, by Theorem 26, there is a unique $\mathcal{A}_{pf}$-morphism $h_\Theta : \mathcal{P}(M/\Theta) \longrightarrow \mathcal{P}(M)/\Sigma_\Theta$ such that $h_\Theta\circ\eta_{M/\Theta} = (\eta_M)_{\Theta,\Sigma_\Theta}$. We now claim that g_Θ and h_Θ are inverse $\mathcal{A}_{pf}$-isomorphisms and, in fact, the unique (iso)morphisms between the arrows $\mathcal{P}(q_\Theta) : \mathcal{P}(M) \twoheadrightarrow \mathcal{P}(M/\Theta)$ and $q_{\Sigma_\Theta} : \mathcal{P}(M) \twoheadrightarrow \mathcal{P}(M)/\Sigma_\Theta$, for we have:

$*$ $\underline{h_\Theta\circ g_\Theta = id}$: Since

$$\begin{aligned} h_\Theta\circ g_\Theta\circ q_{\Sigma_\Theta}\circ\eta_M &= h_\Theta\circ\mathcal{P}(q_\Theta)\circ\eta_M = h_\Theta\circ\eta_{M/\Theta}\circ q_\Theta = (\eta_M)_{\Theta,\Sigma_\Theta}\circ q_\Theta = q_{\Sigma_\Theta}\circ\eta_M \\ &= id\circ q_{\Sigma_\Theta}\circ\eta_M, \end{aligned}$$

and the conclusion follows from the universal property of η_M and the surjectivity of q_{Σ_Θ};

$*$ $\underline{g_\Theta\circ h_\Theta = id}$: Since

$$\begin{aligned} g_\Theta\circ h_\Theta\circ\eta_{M/\Theta}\circ q_\Theta &= g_\Theta\circ(\eta_M)_{\Theta,\Sigma_\Theta}\circ q_\Theta = g_\Theta\circ q_{\Sigma_\Theta}\circ\eta_M = \mathcal{P}(q_\Theta)\circ\eta_M = \eta_{M/\Theta}\circ q_\Theta \\ &= id\circ\eta_{M/\Theta}\circ q_\Theta, \end{aligned}$$

and the conclusion follows from the surjectivity of q_Θ and the universal property of $\eta_{M/\Theta}$, completing the proof of (b). It is clear from the calculations above that g_Θ and h_Θ are inverse isomorphisms between the arrows $\mathcal{P}(q_\Theta) : \mathcal{P}(M) \twoheadrightarrow \mathcal{P}(M/\Theta)$ and $q_{\Sigma_\Theta} : \mathcal{P}(M) \twoheadrightarrow \mathcal{P}(M)/\Sigma_\Theta$, while their uniqueness stems from the fact that $\mathcal{P}(q_\Theta)$ and q_{Σ_Θ} are both surjective. □

4 Concluding Remarks

Now we will suppose a bit more on the elementary class $\mathcal{A}\subseteq$ **L-mod**: that $\mathcal{A}$ is axiomatizable by sentences like $\forall\vec{x}(\varphi(\vec{x})\to\psi(\vec{x}))$ where $\varphi(\vec{x}),\psi(\vec{x})\in[\exists,\wedge,atom(L)]$ or are the negations of atomic L-formulas, in particular, $\mathcal{A}$ is *closed under L-products.* As seen in section 1, profinite structures are pure injective. Given such kind of class $\mathcal{A}$ of L-structures and a saturated family of $\mathcal{A}$-congruences, $\mathfrak{C}$, it is natural consider the subclass $\mathcal{A}_{LG}\subseteq\mathcal{A}$ of (discrete) structures M in $\mathcal{A}$ such that the canonical arrow $\eta_M : M \longrightarrow \mathcal{P}(M)$ is a *pure L-embedding.* This can be rephrased as a *local-global principle*, as follows: M is in $\mathcal{A}_{LG}$ if for all p.p. L-formulas, $\phi(\vec{x})$, and all $\overline{a}\in M^n$,

$$[\text{LG}] \qquad \begin{cases} M\models\phi[\overline{a}] & \Leftrightarrow \quad \mathcal{P}(M)\models\phi[\eta_M(\overline{a})] \\ & \Leftrightarrow \quad \text{For all } C\in\mathfrak{C}(M), \text{ such that } M/C \text{ is finite}, \ \ M/C\models\phi[\overline{a}/C]. \end{cases}$$

The following examples illustrate the principle [LG]:

(a) Boolean Algebras. Every boolean algebra satisfies [LG]. In [**Mrn**], the BA-morphism $\eta_B : B \longrightarrow \mathcal{P}(B)$ is identified with the BA-embedding of B into the clopens of its Stone space. But it follows from Proposition 3.(c) and Sikorsky's Extension Theorem that all BA-monics are pure, since any Boolean algebra is the directed limit of its *finite* subalgebras.

(b) Reduced Special Groups (RSG). By a result in [**GM**], formulated in the dual category of abstract order spaces, there are reduced special group that do not verify [LG] (see also [**Mar**]). In [**AT**] is shown that the subclass of RSGs that satisfy this local-global principle is also an elementary class $\forall\exists$-axiomatizable. In [**Mrn**] it is shown that a weaker formulation of [LG] holds for all RSGs: the morphism $\eta_G\ G \longrightarrow \mathcal{P}(G)$ reflects subforms; in particular, it is a complete embedding and reflects isotropy of forms with coefficients in G.

From the universal property of the profinite hull functor (Theorem 26) and Proposition 3, it follows that

$\mathcal{A}_{LG} \hookrightarrow \mathcal{A}$ is closed under L-isomorphisms, pure L-substructures and L-products, while the complementary subclass, $(\mathcal{A} \setminus \mathcal{A}_{LG})$ is closed under reduced powers. We then pose the following

Problem 1. *Is the class $\mathcal{A}_{LG}$ closed under quotients by elements of $\mathfrak{C}$?*

If L-structures in $\mathcal{A}$ are *L-inhabited* (see Remark 9.(a)) and the answer to the above problem is affirmative, then any reduced product of structures in $\mathcal{A}_{LG}$ is also in $\mathcal{A}_{LG}$. A model-theoretic consequence of this and the closure properties established above shows that $\mathcal{A}_{LG}$ is an L-elementary class *axiomatizable by Horn sentences* (see [**CK**] Theorems 4.1.12 and 6.2.5).

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